

On finite simple groups acting on homology spheres with small fixed point sets

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Abstract. *A finite nonabelian simple group does not admit a free action on a homology sphere, and the only finite simple group which acts on a homology sphere with at most 0-dimensional fixed point sets ("pseudofree action") is the alternating group \mathbb{A}_5 acting on the 2-sphere. Our first main theorem is the finiteness result that there are only finitely many finite simple groups which admit a smooth action on a homology sphere with at most d -dimensional fixed points sets, for a fixed d . We then go on proving that the finite simple groups acting on a homology sphere with at most 1-dimensional fixed point sets are the alternating group \mathbb{A}_5 in dimensions 2, 3 and 5, the linear fractional group $\mathrm{PSL}_2(7)$ in dimension 5, and possibly the unitary group $\mathrm{PSU}_3(3)$ in dimension 5 (we conjecture that it does not admit any action on a homology 5-sphere but cannot exclude it at present). Finally, we discuss the situation for arbitrary finite groups which admit an action on a homology 3-sphere.*

1. Introduction

We are interested in finite groups, and in particular in finite simple groups, which admit a smooth action on an integer or a mod 2 homology sphere. A homology sphere (resp. a mod 2 homology sphere) is a closed manifold with the integer homology of a sphere (resp. the mod 2 homology of a sphere, i.e. with coefficients in the integers mod 2). In the present paper, simple group will always mean *nonabelian* simple group; also, all actions will be smooth (or locally linear), orientation-preserving and faithful.

By [24], [15-17], the only finite simple group which admits an action on a homology 3-sphere is the alternating group \mathbb{A}_5 , and the only finite simple groups acting on a homology 4-sphere are the alternating groups \mathbb{A}_5 and \mathbb{A}_6 . The finite simple groups acting on a homology 5-sphere are considered in [10, Theorem 2] (see Theorem 4 in section 2).

Suppose that a finite nontrivial group G admits a *free* action on an integer homology sphere of dimension n . Since we are considering orientation-preserving actions, by the Lefschetz fixed point theorem this is possible only in odd dimensions; also, G has

periodic cohomology (cf. [4, chapters I.6 and VII.10]), and the class of groups of periodic cohomology is well-known and quite restricted (see [1]); in particular, no finite simple groups occur. A next case which has been considered is that of *pseudo-free actions*, i.e. actions with 0-dimensional fixed point sets; such actions exist only in even dimensions since the fixed point set of any finite cyclic (orientation-preserving) subgroup \mathbb{Z}_p has even codimension (since this is the case for the linear action induced on the tangent space of a fixed point). For this case it is easy to see that again only the groups with periodic cohomology occur, plus the finite groups acting on the 2-sphere (see the Remark at the end of section 2); in particular, the only finite simple group which occurs is the alternating group \mathbb{A}_5 acting on S^2 .

Thus one is led to consider less restrictive conditions on fixed point sets. For an integer $d \geq -1$, we say that a finite group acts *with at most d -dimensional fixed points sets* if the fixed point of each nontrivial element has dimension at most d (where $d = -1$ stands for empty fixed point set). Our first main result is the following:

Theorem 1. *For a fixed d , there are only finitely many finite simple groups which admit an action on a homology sphere with at most d -dimensional fixed point sets.*

We believe that such a finiteness result does not hold for actions of finite simple groups on mod 2 homology spheres. For example, it is likely that all groups $\mathrm{PSL}_2(p)$, p prime, admit an action already on a mod 2 homology 3-sphere, that is in dimension three; examples of such actions for various small values of p are given in [26] but the problem remains open in general (see also the survey [25]).

Next we consider the case $d = 1$:

Theorem 2. *The finite simple groups which act with at most 1-dimensional fixed point sets on a homology sphere are the alternating group \mathbb{A}_5 in dimensions 2, 3 and 5, the linear fractional group $\mathrm{PSL}_2(7)$ in dimension 5, and possibly the unitary group $\mathrm{PSU}_3(3)$ in dimension 5.*

We note that the groups \mathbb{A}_5 and $\mathrm{PSL}_2(7)$ admit linear actions with at most 1-dimensional fixed point sets on spheres of the indicated dimensions (see the proof of Theorem 2). The unitary group $\mathrm{PSU}_3(3)$ has a linear action on S^6 (with at most 2-dimensional fixed point sets); we conjecture that it does not admit any action on a homology 5-sphere but cannot exclude it at present.

The proofs of Theorems 1 and 2 are based also on a part of the following theorem which collects some consequences of the Borel-formula for actions of an elementary abelian p -group ([2, Theorem XIII.2.3]), in combination with some deep results from the theory of finite simple groups.

Theorem 3. i) Let G be a finite group which admits an action on a mod 2 homology n -sphere such that involutions have at most d -dimensional fixed points sets. Suppose that G has a subgroup isomorphic to the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Then

$$n \leq 3d + 2;$$

in particular, this holds if G is a nonabelian simple group.

ii) Suppose that $n = 3d + 2$. If G is a nonabelian simple group then G has 2-rank at most two (i.e., G has no subgroups $(\mathbb{Z}_2)^3$) and is one of the following groups (where q denotes an odd prime power):

$$\mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), \mathrm{PSU}_3(4), \mathbb{A}_7, \text{ the Mathieu group } M_{11}.$$

iii) Suppose that $n > 3d + 2$. Then G has no subgroup $(\mathbb{Z}_2)^2$, and hence 2-periodic cohomology. Moreover, if G is nonsolvable then it has the following structure: denoting by $\mathcal{O}(G)$ the maximal normal subgroup of odd order of G , the factor group $G/\mathcal{O}(G)$ contains a normal subgroup of odd index which is isomorphic to

$$\mathrm{SL}_2(q), \mathrm{TL}_2(q) \text{ or } \hat{\mathbb{A}}_7.$$

Here $\hat{\mathbb{A}}_7$ denotes the unique perfect central extension of the alternating group \mathbb{A}_7 , with center of order two, and $\mathrm{TL}_2(q)$ is the 2-fold extension of $\mathrm{SL}_2(q)$ with a unique involution whose quotient group is isomorphic to $\mathrm{PGL}_2(q)$ (see [1, chapter IV.6]).

We refer to [4, Theorem VI.9.7] and [23] for the notion of a p -periodic group. We note also that a finite group admits a free action on a mod 2 homology sphere if and only if it is 2-periodic and has a unique involution, see [20].

In section 3 we consider finite groups acting on a homology 3-sphere. We note that, by the recent geometrization of finite group-actions on 3-manifolds due to Thurston and Perelman, the finite groups which admit an action on the 3-sphere S^3 are exactly the finite subgroups of the orthogonal group $\mathrm{O}(4)$. This is no longer true for finite groups which admit an action on an arbitrary homology 3-sphere; however the classification of such groups remains open and appears to be difficult (even for quite easy types of finite groups as in Question 2 i) of section 3).

2. Proofs

We start with the *Proof of Theorem 3*. Let G be a finite group acting on a mod 2 homology n -sphere with at most d -dimensional fixed point sets; suppose that G has a subgroup $(\mathbb{Z}_2)^2$. We note that in particular every finite nonabelian simple group has a subgroup $(\mathbb{Z}_2)^2$: in fact, if a finite group has no subgroup $(\mathbb{Z}_2)^2$ then, by a theorem of Burnside ([22, 4.4.3] or [4, Theorem IV.4.3]), a Sylow 2-subgroup is either cyclic or

generalized quaternion, but by [22, p. 144, Corollary 1] and [22, p. 306, Example 3] a Sylow 2-subgroup of a finite simple group cannot be cyclic or generalized quaternion.

By Smith fixed point theory ([3], [21]), the fixed point set of an orientation-preserving periodic map of prime order p of a mod p homology sphere is again a mod p homology sphere, of even codimension. A basic tool for actions of an elementary abelian p -group $A \cong (\mathbb{Z}_p)^m$ on a mod p homology n -sphere is then the following Borel formula ([2, Theorem XIII.2.3]):

$$n - r = \sum_H (r(H) - r)$$

where the sum is taken over all subgroups H of index p of A , $r(H)$ denotes the dimension of the fixed point set of a given subgroup H and $r = r(A)$ the dimension of the fixed point set of A (equal to -1 if the fixed point set is empty).

Applying the Borel formula to a subgroup $A \cong (\mathbb{Z}_2)^2$ of G and using the fact that $-1 \leq r(A), r(H) \leq d$, we obtain the inequality

$$n = \sum_H r(H) - 2r \leq 3d + 2,$$

proving part i) of Theorem 3.

Suppose that $n = 3d + 2$. Then $r(H) = d$ for each of the three \mathbb{Z}_2 subgroups of A , and $r = r(A) = -1$. Suppose that G has a subgroup $B \cong (\mathbb{Z}_2)^3$. Then B has exactly seven subgroups $A \cong (\mathbb{Z}_2)^2$ of index two, $r(A) = -1$ for each of these by the above, and in particular also $r(B) = -1$. Applying the Borel formula to B now we obtain a contradiction ($n = -7 - 6r = -13$). So G does not have an elementary abelian subgroup $(\mathbb{Z}_2)^3$ of rank three and has 2-rank equal to two. By a fundamental result in the classification of the finite simple groups ([9]), the finite simple groups of 2-rank two are exactly the groups listed in Theorem 3 ii).

Finally, suppose that $n > 3d + 2$. Then as before G has no subgroups $(\mathbb{Z}_2)^2$ and hence, by [4, Theorem VI.9.7], G has 2-periodic cohomology.

Suppose that G is nonsolvable. By the Feit-Thompson theorem, a Sylow 2-subgroup S of G is nontrivial. Since the finite 2-group S has nontrivial center and no subgroups $(\mathbb{Z}_2)^2$, S has a unique involution. By the theorem of Burnside above, a finite 2-group with a unique involution is either cyclic or generalized quaternion. Since the Sylow 2-subgroup of a nonsolvable group cannot be cyclic ([22, chapter 5.2, Corollary 2]), S is a generalized quaternion group. The structure of the finite nonsolvable groups with a generalized quaternion Sylow 2-subgroup is given in [22, chapter 6, Theorem 8.7], and the version given in Theorem 3 is an elaboration of this as in [25, Theorem 5].

This completes the proof of Theorem 3.

Theorem 1 is now a consequence of Theorem 1 i) and of [10, Theorem 1] stating that for each dimension n there are only finitely many finite simple groups which admit an action on a homology n -sphere.

For the *Proof of Theorem 2*, let G be a finite simple group acting on a homology n -sphere with at most 1-dimensional fixed point sets; by Theorem 3 i), $n \leq 5$.

By [24], the only finite simple group acting on a homology 3-sphere is \mathbb{A}_5 .

As noted above, by Smith fixed point theory the fixed point set of an orientation-preserving periodic map of prime order p of a mod p homology sphere is again a mod p homology sphere of even codimension. Now the case $n = 4$ is excluded by the Borel formula applied to a subgroup $(\mathbb{Z}_2)^2$ of G (since the hypothesis of at most 1-dimensional fixed point sets implies that $r(\mathbb{Z}_2) = 0$ in dimension 4).

So we are left with the case $n = 5$: suppose that G acts on a homology 5-sphere. By Theorem 3 ii), G has 2-rank two and is one of the following groups:

$$\mathrm{PSL}_2(q), \mathrm{PSL}_3(q), \mathrm{PSU}_3(q), \mathrm{PSU}_3(4), \mathbb{A}_7, \text{ the Mathieu group } M_{11}.$$

For the proof of Theorem 2 we have to exclude all of these groups except $\mathbb{A}_5 \cong \mathrm{PSL}_2(5)$, $\mathrm{PSL}_2(7)$ and $\mathrm{PSU}_3(q)$. This is based on the following result from [10] (resp. on some part of its proof):

Theorem 4. ([10, Theorem 2]) *The finite simple groups which admit an action on a homology 5-sphere are the following:*

$$\mathbb{A}_5 \cong \mathrm{PSL}_2(5), \mathbb{A}_6 \cong \mathrm{PSL}_2(9), \mathbb{A}_7, \mathrm{PSL}_2(7), \mathrm{PSU}_4(2), \text{ possibly } \mathrm{PSU}_3(3).$$

With the exception of the unitary group $\mathrm{PSU}_3(3)$ (which admits a linear action on S^6) these are exactly the finite simple groups which admit a linear action on S^5 . We note that the proof of [10, Theorem 2] is on the basis of the full classification of the finite simple groups; in our situation the proof is much shorter since we have to consider only the quite restricted list of the finite simple groups of 2-rank two.

Note that for the proof of Theorem 2 we still have to exclude that the alternating groups \mathbb{A}_6 and \mathbb{A}_7 from the list in Theorem 4 admit an action on a homology 5-sphere with at most 1-dimensional fixed point sets. Suppose that \mathbb{A}_6 admits such an action. We consider an elementary abelian subgroup $(\mathbb{Z}_3)^2$ of \mathbb{A}_6 generated by two disjoint cycles of length three, with four subgroups \mathbb{Z}_3 ; note that the four subgroups \mathbb{Z}_3 of $(\mathbb{Z}_3)^2$ are conjugate in pairs in \mathbb{A}_6 (two are generated by a 3-cycle, the other two by a product of two 3-cycles). By the Borel formula, $n - r = 5 - r = \sum_i (r(H_i) - r)$, or

$$5 + 3r = r(H_1) + r(H_2) + r(H_3) + r(H_4)$$

where the H_i denote the four subgroups \mathbb{Z}_3 of $(\mathbb{Z}_3)^2$. By our assumption of at most 1-dimensional fixed point sets, we have that $r(H_i) \leq 1$, hence also $r \leq 1$ and $5 + 3r \leq 4$. This excludes the possibilities $r = 0$ and $r = 1$. Suppose that $r = -1$. Then the only solution of the Borel formula is $5 + 3r = 2 = r(H_1) + r(H_2) + r(H_3) + r(H_4) = 1 + 1 + 1 - 1$ (note that $r(H_i) \neq 0$ since the fixed point set of each H_i has even codimension); however also this solution is not possible since the four subgroups H_i of $(\mathbb{Z}_3)^2$ are conjugate in pairs.

This excludes the groups \mathbb{A}_6 and \mathbb{A}_7 , and the only finite simple groups which remain from the list in Theorem 4 are \mathbb{A}_5 , $\mathrm{PSL}_2(7)$ and $\mathrm{PSU}_3(3)$. The dodecahedral group \mathbb{A}_5 acts on S^2 with 0-dimensional fixed point sets S^0 , and it has two linear actions on S^3 : one is the suspension of the action on S^2 , with two global fixed points, the other one is the restriction to S^3 of its irreducible 4-dimensional real representation (a summand of the standard 5-dimensional representation of \mathbb{A}_5 by permutation of coordinates of \mathbb{R}^5). A linear action of \mathbb{A}_5 on S^5 is obtained by considering $S^5 \cong S^2 * S^2$ as the join of two 2-spheres and by taking also the join of two actions of \mathbb{A}_5 on S^2 , with fixed point sets $S^0 * S^0 \cong S^1$ (or equivalently by restricting to S^5 the direct sum of two irreducible 3-dimensional real representations). The group $\mathrm{PSL}_2(7)$ has an irreducible 3-dimensional complex representation, and the restriction of the corresponding 6-dimensional (reducible) real representation to S^5 has 1-dimensional fixed point sets (see the character tables in [5]; note that $\mathrm{PSL}_2(7)$ has also an irreducible 6-dimensional real representation whose restriction, however, has also 3-dimensional fixed point sets).

This completes the proof of Theorem 2.

Remark. Suppose that a finite nontrivial group G acts orientation-preservingly and pseudofreely (i.e., with at most 0-dimensional fixed point sets) on a homology n -sphere. Applying the Borel formula, G has no subgroup $(\mathbb{Z}_2)^2$ if $n > 2$, and no subgroup $(\mathbb{Z}_p)^2$ for odd primes p . So, if $n > 2$, every abelian subgroup of G is cyclic and hence G has periodic cohomology (see [4, Proposition VI.9.3]). The groups of periodic cohomology are well-known (see e.g. [1]). If such a group has in addition a unique involution then it is known as an application of high-dimensional surgery theory that it admits a free action on a sphere of odd dimension ([14]); by suspending such a free action one obtains a pseudofree action with exactly two global fixed points on a sphere of even dimension. Thus the finite groups which admit a pseudofree action on some homology sphere are exactly the finite groups acting on S^2 , plus the groups of periodic cohomology with a unique involution. Kulkarni has shown ([12, Theorem 7.4]) that, with the only exception of maybe dihedral groups, every pseudofree action on a homology sphere of dimension at least three has exactly two global fixed points such that the action on the complement of these two fixed points is free ("semifree action").

3. The situation in dimension three

By the recent geometrization of finite group-actions on 3-manifolds due to Thurston and Perelman, every finite group of diffeomorphisms of S^3 is conjugate to a subgroup of the orthogonal group $O(4)$; in particular, the finite groups occurring are exactly the finite subgroups of the orthogonal group $O(4)$. The finite groups which admit an action on an arbitrary homology 3-sphere are discussed in [27]; a complete classification of these groups is not known. We consider first the case of free actions.

If a finite group G admits a *free* action on a homology 3-sphere then G has periodic cohomology of period four and a unique involution; a list of such groups is given in [19], together with the subclass of all finite groups which admit a free, *linear* action on the 3-sphere. By [13] there remains one class of groups $Q(8a, b, c)$ in [19] which do not admit a free, linear action on S^3 but for which the existence of a free action on a homology 3-sphere remains open, in general. By [18], some of the groups $Q(8a, b, c)$ admit a free action on a homology 3-sphere and some others do not, but the exact classification remains open (see also the discussion in [11, Problem 3.37 Update A (p.173)]).

The group $Q(8a, b, c)$ has a presentation

$$\langle x, y, z \mid x^2 = (xy)^2 = y^{2a}, z^{bc} = 1, xzx^{-1} = z^r, yzy^{-1} = z^{-1} \rangle,$$

for relatively coprime positive integers $8a, b$ and c such that either a is odd and $a > b > c \geq 1$, or $a \geq 2$ is even and $b > c \geq 1$; also, $r \equiv -1 \pmod{b}$ and $r \equiv +1 \pmod{c}$. Note that $Q(8a, b, c)$ is a semidirect product $\mathbb{Z}_{bc} \rtimes Q(8a)$, with normal subgroup $\mathbb{Z}_{bc} \cong \mathbb{Z}_b \times \mathbb{Z}_c$ generated by z , and factor group the generalized quaternion or binary dihedral subgroup $Q(8a) \cong Q(8a, 1, 1)$ of order $8a$ generated by x and y . See also [6, section 7] for a description of these groups and various inclusions between them.

We note that a group $Q(8a, b, c)$ does not admit a free action on a homology 3-sphere if a is even ([13], [6]). If a is odd then $Q(8a) \cong \mathbb{Z}_a \rtimes Q(8)$, and hence $Q(8a, b, c)$ is an extension of $\mathbb{Z}_{abc} \cong \mathbb{Z}_a \times \mathbb{Z}_b \times \mathbb{Z}_c$ by the quaternion group $Q(8) = \{\pm 1, \pm i, \pm j, \pm k\}$ such that i, j, k acts trivially on $\mathbb{Z}_a, \mathbb{Z}_b, \mathbb{Z}_c$, respectively, and in a dihedral way on the other two.

Concerning *nonfree* actions of the groups $Q(8a, b, c)$, we note the following:

Proposition 1. *A group $Q(8a, b, c)$ does not admit a nonfree action on a homology 3-sphere (orientation-preserving or not).*

Proof. Suppose that $G = Q(8a, b, c)$ acts orientation-preservingly on a homology 3-sphere M . The unique involution $h = x^2 = (xy)^2 = y^{2a}$ of G is central in G ; by Smith fixed point theory the fixed point set of h has even codimension and is either empty or a 1-sphere S^1 in M (see e.g. [3]). Suppose that h has nonempty fixed point set S^1 ;

note that S^1 is invariant under the action of G . We note that, if a finite orientation-preserving group leaves invariant a 1-sphere S^1 in a 3-manifold then G is isomorphic to a subgroup of a semidirect product $(\mathbb{Z}_m \times \mathbb{Z}_n) \rtimes \mathbb{Z}_2$, with a dihedral action of \mathbb{Z}_2 on $\mathbb{Z}_m \times \mathbb{Z}_n$ (here \mathbb{Z}_2 acts as a reflection or strong inversion on S^1 whereas $\mathbb{Z}_m \times \mathbb{Z}_n$ acts by rotations about and along S^1). Since clearly $G = Q(8a, b, c)$ is not of this type, the unique involution h of G has to act freely, and hence also every nontrivial element in G of even order.

Next suppose that some nontrivial element $g \in G$ of odd prime order has nonempty fixed point set S^1 , acting as a rotation about S^1 ; we can assume that g is an element of one of the subgroups $\mathbb{Z}_b \times \mathbb{Z}_c$ or $Q(8a) \cong Q(8a, 1, 1)$ of G . In each case some element u of even order in the generalized quaternion group $Q(8a)$ acts dihedrally on g (i.e., $ugu^{-1} = g^{-1}$). Since u has no fixed points, it acts as a rotation along and about S^1 . But then u commutes with the rotation g about S^1 which is a contradiction.

Now suppose that some element of G reverses the orientation of M . Since the order bc of z is odd, x or xy are orientation-reversing; we assume that x is orientation-reversing (the case of xy is analogous). By Smith fixed point theory, the fixed point set of x has odd codimension and is either a 0-sphere (two points) or a 2-sphere. Then the fixed point set of the central involution $h = x^2$ is also nonempty and hence a 1-sphere S^1 (of even codimension), the fixed point set of x is a 0-sphere $S^0 \subset S^1$, and x acts as a reflection (strong inversion) on S^1 .

If both x and xy reverse the orientation of M then also xy acts as a reflection on S^1 , and y is orientation-preserving and acts as a rotation about and along S^1 . But also the subgroup \mathbb{Z}_{bc} generated by z acts as a group of rotations about and along S^1 , so y and z commute; this is a contradiction since y acts dihedrally on z .

So xy acts orientation-preservingly and its fixed point set is either empty or S^1 . If xy has empty fixed point set then it acts by rotations about and along S^1 , and hence commutes with y^2 (of order $2a \geq 4$) which acts also by rotations about and along S^1 . This is a contradiction since x and hence xy acts dihedrally on y and y^2 (specifically, $(xy)^2 = xyxy = h$ implies that $xyx^{-1} = hy^{-1}x^{-2} = hy^{-1}h^{-1} = y^{-1}$). If xy fixes S^1 instead then it acts by rotations about S^1 and commutes again with y^2 , so we obtain the same contradiction as before.

So there are no orientation-reversing actions of G on a homology 3-sphere. This completes the proof of Proposition 1.

By results of Milgram [18], some of the groups $Q(8a, b, c)$ admit a free action on a homology 3-sphere. By the geometrization of 3-manifolds with finite fundamental group, none of the groups $Q(8a, b, c)$ admits a free action on S^3 (since, by [19], they do not admit a free, linear action on S^3). By Proposition 1, they also don't admit nonfree actions on S^3 (alternatively, considering orthogonal actions, one can confront them

with the list of the finite subgroups of $O(4)$ in [8], see also [7] for the geometry of their quotient orbifolds in the orientation-preserving case). Summarizing, the following holds:

Proposition 2. *The class of finite groups which admit an action on a homology 3-sphere is strictly larger than the class of finite groups which admit an action on S^3 (or the class of finite subgroups of $SO(4)$).*

There arises naturally the question of how big the difference is between the classes of groups in Proposition 2: do there occur other groups than the Milnor groups $Q(8a, b, c)$? If a finite group admits a free action on a homology 3-sphere but not on S^3 then it is in fact one of the Milnor groups $Q(8a, b, c)$, with a odd ([13]), so any other such group would admit only nonfree actions on a homology 3-sphere.

Question 1. i) Does there exist a finite group which admits a nonfree, orientation-preserving action on a homology 3-sphere but is not isomorphic to a subgroup of the orthogonal group $SO(4)$?

ii) Is there a finite group with an orientation-reversing action on a homology 3-sphere which is not isomorphic to a subgroup of $O(4)$?

In the following, concentrating on the orientation-preserving case, we discuss some natural candidates. It is shown in [27] that the finite nonsolvable groups which admit an orientation-preserving action on a homology 3-sphere are exactly the finite nonsolvable subgroups of the orthogonal group $SO(4) \cong S^3 \times_{\mathbb{Z}_2} S^3$ (the central product of two copies of the unit quaternions), plus possibly two other classes of groups:

- the central products

$$\mathbb{A}_5^* \times_{\mathbb{Z}_2} Q(8a, b, c)$$

where a is odd and \mathbb{A}_5^* denotes the binary dodecahedral group;

- their subgroups

$$\mathbb{A}_5^* \times_{\mathbb{Z}_2} (\mathbb{D}_{4a}^* \times \mathbb{Z}_b)$$

where $\mathbb{D}_{4a}^* \cong Q(4a) \cong \mathbb{Z}_a \rtimes \mathbb{Z}_4$ denotes the binary dihedral or generalized quaternion group of order $4a$.

In turn these have subgroups

$$\mathbb{D}_8^* \times_{\mathbb{Z}_2} (\mathbb{D}_{4a}^* \times \mathbb{Z}_b)$$

which do not act freely on a homology 3-sphere (since they have a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$).

Lemma. *For odd coprime integers $a, b \geq 3$, the group $G = \mathbb{D}_8^* \times_{\mathbb{Z}_2} \mathbb{D}_{4a}^* \times \mathbb{Z}_b$ does not admit an orientation-preserving, linear action on S^3 .*

Proof. Suppose that G is a subgroup of the orthogonal group $\mathrm{SO}(4) \cong S^3 \times_{\mathbb{Z}_2} S^3$. The finite subgroups of the unit quaternions S^3 are cyclic, binary dihedral or binary polyhedral groups. The two projections of the subgroup \mathbb{D}_8^* of G to the first and second factor of $S^3 \times_{\mathbb{Z}_2} S^3$ are cyclic or binary dihedral groups; since \mathbb{D}_8^* is nonabelian, one of the two projections, say the first one, has to be a binary dihedral group. Then, since the projections of the subgroups \mathbb{D}_8^* and \mathbb{D}_{4a}^* of G commute elementwise, the projection of \mathbb{D}_{4a}^* to the second factor of $S^3 \times_{\mathbb{Z}_2} S^3$ has to be a binary dihedral group. But then at least one of the two projections of the cyclic subgroup \mathbb{Z}_b of G (any nontrivial one) does not commute elementwise with either the binary dihedral projection of \mathbb{D}_8^* or that of \mathbb{D}_{4a}^* . This contradiction completes the proof of the Lemma.

Question 2. i) For odd, coprime integers $a, b > 1$, does

$$\mathbb{D}_8^* \times_{\mathbb{Z}_2} \mathbb{D}_{4a}^* \times \mathbb{Z}_b$$

admit an orientation-preserving action on a homology 3-sphere? (If a is even then there is no such action by [27, Lemma].)

ii) Does the central product

$$\mathbb{Z}_4 \times_{\mathbb{Z}_2} Q(8a, b, c)$$

admit an action on some homology 3-sphere (assuming that $Q(8a, b, c)$ does)?

Note that these groups do not act freely on a homology 3-sphere (since they have a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$) and are not isomorphic to a subgroup of $\mathrm{SO}(4)$ (by the Lemma, and since $Q(8a, b, c)$ is not).

If the answer to i) is negative then by [27] the class of the finite nonsolvable groups which admit an orientation-preserving action on a homology 3-sphere coincides with the class of the finite nonsolvable subgroups of the orthogonal group $\mathrm{SO}(4)$. On the other hand, if a group in i) or ii) admits such an action then this would give a first example of a finite group which admits a nonfree, orientation-preserving action on a homology 3-sphere but which is not isomorphic to a subgroup of $\mathrm{SO}(4)$ (and in case i) independently of the quite difficult Milnor groups $Q(8a, b, c)$).

Finally, considering also the case of mod 2 homology 3-spheres, we close with the following:

Conjecture. Each linear fractional group $\mathrm{PSL}_2(p)$, p prime, admits an action on a mod 2 homology 3-sphere.

By [15] or [16], these are exactly the candidates among the finite nonabelian simple groups which possibly admit an action on a mod 2 homology 3-sphere. Examples of such actions for various small values of p are given in [26]; see [15], [16] or the survey

[25] for a partial classification of the finite nonsolvable groups which admit an action on a mod 2 homology 3-sphere.

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